

# Reformulation of the Li criterion for the Selberg class

Kamel Mazhouda

Faculty of Sciences of Monastir, Department of Mathematics,  
5000 Monastir, Tunisia  
E-mail: Kamel.Mazhouda@fsm.rnu.tn

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## Abstract

Let  $F$  be a function in the Selberg class  $\mathcal{S}$  and  $a$  be a real number not equal to  $1/2$ . Consider the sum

$$\lambda_F(n, a) = \sum_{\rho} \left[ 1 - \left( \frac{\rho - a}{\rho + a - 1} \right)^n \right],$$

where  $\rho$  runs over the non-trivial zeros of  $F$ . In this paper, we prove that the Riemann hypothesis is equivalent to the positivity of the "modified Li coefficient"  $\lambda_F(n, a)$ , for  $n = 1, 2, \dots$  and  $a < 1/2$ . Furthermore, we give an explicit arithmetic and asymptotic formula of these coefficients.

## 1 Introduction

The Riemann hypothesis is the subject of several studies and research papers. Most of them provide new reformulations and numerical evidence for this hypothesis. In the literature, there exists various formulations of the Riemann hypothesis. The Li criterion for the Riemann hypothesis (see. [10]) is a necessary and sufficient condition that the sequence

$$\lambda_n = \sum_{\rho} \left[ 1 - \left( 1 - \frac{1}{\rho} \right)^n \right]$$

is non-negative for all  $n \in \mathbb{N}$  and where  $\rho$  runs over the non-trivial zeros of  $\zeta(s)$ . This criterion holds for a large class of Dirichlet series so called the Selberg class

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as given in [15, 16]. More recently, Omar and Bouanani [14] extended the Li criterion for function fields and established an explicit and asymptotic formula for the Li coefficients.

The Selberg class  $\mathcal{S}$  [23] consists of Dirichlet series

$$F(s) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^s}, \quad \operatorname{Re}(s) > 1$$

satisfying the following hypothesis.

- **Analytic continuation:** there exists a non negative integer  $m$  such  $(s-1)^m F(s)$  is an entire function of finite order. We denote by  $m_F$  the smallest integer  $m$  which satisfies this condition;
- **Functional equation:** for  $1 \leq j \leq r$ , there are positive real numbers  $Q_F$ ,  $\lambda_j$  and there are complex numbers  $\mu_j$ ,  $\omega$  with  $\operatorname{Re}(\mu_j) \geq 0$  and  $|\omega| = 1$ , such that

$$\phi_F(s) = \overline{\omega \phi_F(1 - \bar{s})}$$

where

$$\phi_F(s) = F(s) Q_F^s \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j);$$

- **Ramanujan hypothesis:**  $a(n) = O(n^\epsilon)$ ;
- **Euler product:**  $F(s)$  satisfies

$$F(s) = \prod_p \exp \left( \sum_{k=1}^{+\infty} \frac{b(p^k)}{p^{ks}} \right)$$

with suitable coefficients  $b(p^k)$  satisfying  $b(p^k) = O(p^{k\theta})$  for some  $\theta < \frac{1}{2}$ .

It is expected that for every function in the Selberg class the analogue of the Riemann hypothesis holds, i.e, that all non trivial (non-real) zeros lie on the critical line  $\operatorname{Re}(s) = \frac{1}{2}$ . The degree of  $F \in \mathcal{S}$  is defined by

$$d_F = 2 \sum_{j=1}^r \lambda_j.$$

The logarithmic derivative of  $F(s)$  has also the Dirichlet series expression

$$-\frac{F'}{F}(s) = \sum_{n=1}^{+\infty} \Lambda_F(n) n^{-s}, \quad \operatorname{Re}(s) > 1,$$

where  $\Lambda_F(n) = b(n) \log n$  is the generalized von Mangoldt function. If  $N_F(T)$  counts the number of zeros of  $F(s) \in \mathcal{S}$  in the rectangle  $0 \leq \operatorname{Re}(s) \leq 1$ ,  $0 < \operatorname{Im}(s) \leq T$  (according to multiplicities) one can show by standard contour integration the formula

$$N_F(T) = \frac{d_F}{2\pi} T \log T + c_1 T + O(\log T), \quad (1)$$

where

$$c_1 = \frac{1}{2\pi} (\log q_F - d_F (\log(2\pi) + 1))$$

and

$$q_F = \frac{(2\pi)^{d_F} Q_F^2}{\prod_{j=1}^r \lambda_j^{-2\lambda_j}}$$

in analogy to the Riemann-von Mangoldt formula for Riemann's zeta-function  $\zeta(s)$ , the prototype of an element in  $\mathcal{S}$ . For more details concerning the Selberg class we refer to the surveys of Kaczorowski [7] and Perelli [22].

## 2 Review on the Li criterion for the Selberg class

Let  $F$  be a function in the Selberg class non-vanishing at  $s = 1$  and let us define the xi-function  $\xi_F(s)$  by

$$\xi_F(s) = s^{m_F} (s-1)^{m_F} \phi_F(s).$$

The function  $\xi_F(s)$  satisfies the functional equation

$$\xi_F(s) = \overline{\omega \xi_F(1 - \overline{s})}.$$

The function  $\xi_F$  is an entire function of order 1. Therefore, the Hadamard factorization theorem implies that the function  $\xi_F(s)$  possesses a representation as the product over its zeros

$$\xi_F(s) = \xi_F(0) e^{b_F s} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}},$$

where  $b_F = \frac{\xi'_F}{\xi_F}(0)$ . Inserting the following test function (firstly considered by K. Barner [2])

$$G_s(x) = \begin{cases} 0, & x > 0, \\ 1/2, & x = 0, \\ e^{(s-1/2)x}, & x < 0 \end{cases} \quad s \in \mathbb{C}, \quad \operatorname{Re}(s) > 1$$

into the Weil explicit formula [16, Proposition], we obtain for  $F(s) \in \mathcal{S}$  non-vanishing at  $s = 0$  and for all  $s \in \mathbb{C}$  different from zeros of  $\xi_F(s)$

$$\frac{\xi'_F}{\xi_F}(s) = \lim_{T \rightarrow \infty} \sum_{|Im(\rho)| \leq T} \frac{\operatorname{ord} \rho}{s - \rho}. \quad (2)$$

Hence, the function  $\xi_F(s)$  can be written as

$$\xi_F(s) = \xi_F(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right), \quad (3)$$

where the product is over all zeros of  $\xi_F(s)$  in the order given by  $|Im(\rho)| < T$  for  $T \rightarrow \infty$ .

Let  $\lambda_F(n)$ ,  $n \in \mathbb{Z}$ , be a sequence of numbers defined by a sum over the non-trivial zeros of  $F(s)$  as

$$\lambda_F(n) = \sum_{\rho} \left[1 - \left(1 - \frac{1}{\rho}\right)^n\right]$$

where the sum over  $\rho$  is

$$\sum_{\rho} = \lim_{T \rightarrow \infty} \sum_{|Im \rho| \leq T}.$$

These coefficients are well defined, indeed: the function  $\xi_F$  is an entire function of order one, hence the series  $\sum_{\rho} \frac{1}{\rho^k}$  converges absolutely for every integer  $k \geq 2$ . From (2) and (3), one has

$$\sum_{\rho} \frac{1}{\rho} = \lim_{T \rightarrow \infty} \sum_{|Im \rho| \leq T} \frac{1}{\rho},$$

then the series

$$\lambda_F(n) = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \sum_{\rho} \frac{1}{\rho^k}$$

exists for every positive integer  $n$ . Let  $\mathcal{Z}$  the multi-set of zeros of  $\xi_F(s)$  (counted with multiplicity). The multi-set  $\mathcal{Z}$  is invariant under the map  $\rho \mapsto 1 - \bar{\rho}$ . We have

$$1 - \left(1 - \frac{1}{\rho}\right)^{-n} = 1 - \left(\frac{\rho-1}{\rho}\right)^{-n} = 1 - \left(\frac{-\rho}{1-\rho}\right)^{-n} = 1 - \overline{\left(1 - \frac{1}{1-\bar{\rho}}\right)^n}$$

and this gives the symmetry  $\lambda_F(-n) = \overline{\lambda_F(n)}$ . This proves that the  $\lambda_F(n)$  exists for all integer  $n$ . Since  $\xi_F(s)$  is an entire function of order 1, and its zeros lie in the critical strip  $0 \leq \operatorname{Re}(s) \leq 1$ , we also obtain that the series

$$\sum_{\rho} \frac{1 + |\operatorname{Re}(\rho)|}{(1 + |\rho|)^2}$$

is convergent. Now the application of the Lagarias lemma [9, Lamma 1] to the multi-set  $\mathcal{Z}$  of non-trivial zeros of  $F(s)$ , the series

$$\operatorname{Re} \lambda_F(n) = \sum_{\rho} \operatorname{Re} \left[ 1 - \left(1 - \frac{1}{\rho}\right)^n \right]$$

converges absolutely for all integer  $n$ . These coefficients are expressible in terms of power-series coefficients of functions constructed from the  $\xi_F$ -function. For  $n \leq -1$ , the Li coefficients  $\lambda_F(n)$  correspond to the following Taylor expansion at the point  $s = 1$

$$\frac{d}{dz} \log \xi_F \left( \frac{1}{1-z} \right) = \sum_{n=0}^{+\infty} \lambda_F(-n-1) z^n$$

and for  $n \geq 1$ , they correspond to the Taylor expression at  $s = 0$

$$\frac{d}{dz} \log \xi_F \left( \frac{-z}{1-z} \right) = \sum_{n=0}^{+\infty} \lambda_F(n+1) z^n.$$

Using [3, Corollary 1], we get the following generalization of the Li criterion for the Riemann hypothesis.

**Theorem 1.** *Let  $F(s)$  be a function in the Selberg class  $\mathcal{S}$  non-vanishing at  $s = 1$ . Then, all non-trivial zeros of  $F(s)$  lie on the line  $\operatorname{Re}(s) = 1/2$  if and only if  $\operatorname{Re}(\lambda_F(n)) > 0$  for  $n = 1, 2, \dots$*

Under the same hypothesis of Theorem 1, the Riemann hypothesis is also equivalent to each of the two following conditions (a) or (b):

(a) For each  $\epsilon > 0$ , there is a positive constant  $c(\epsilon)$  such that

$$\operatorname{Re}(\lambda_F(n)) \geq -c(\epsilon)e^{\epsilon n} \text{ for all } n \geq 1$$

(b) The Li coefficients  $\lambda_F(n)$  satisfy

$$\lim_{n \rightarrow \infty} |\lambda_F(n)|^{1/n} \leq 1.$$

The proof is the same as in [9, Theorem 2.2]. Next, we recall the following explicit formula for the coefficients  $\lambda_F(n)$ . Let consider the following hypothesis:  $\mathcal{H}$  **there exists a constant  $c > 0$  such that  $F(s)$  is non-vanishing in the region:**

$$\left\{ s = \sigma + it; \sigma \geq 1 - \frac{c}{\log(Q_F + 1 + |t|)} \right\}.$$

**Theorem 2.** *Let  $F(s)$  be a function in the Selberg class  $\mathcal{S}$  satisfying  $\mathcal{H}$ . Then, we have*

$$\begin{aligned} \lambda_F(-n) &= m_F + n \left( \log Q_F - \frac{d_F}{2} \gamma \right) \\ &- \sum_{l=1}^n \binom{l}{l} \frac{(-1)^{l-1}}{(l-1)!} \lim_{X \rightarrow +\infty} \left\{ \sum_{k \leq X} \frac{\Lambda_F(k)}{k} (\log k)^{l-1} - \frac{m_F}{l} (\log X)^l \right\} \\ &+ n \sum_{j=1}^r \lambda_j \left( -\frac{1}{\lambda_j + \mu_j} + \sum_{l=1}^{+\infty} \frac{\lambda_j + \mu_j}{l(l + \lambda_j + \mu_j)} \right) \\ &+ \sum_{j=1}^r \sum_{k=2}^n \binom{n}{k} (-\lambda_j)^k \sum_{l=0}^{+\infty} \left( \frac{1}{l + \lambda_j + \mu_j} \right)^k, \end{aligned} \tag{4}$$

where  $\gamma$  is the Euler constant.

**Remark.** The class of functions from  $\mathcal{S}$  satisfying  $\mathcal{H}$  is a sub-class of the class considered by Smajlovic [26] which will be noted by  $\tilde{\mathcal{S}}$ . Indeed, first recall that the Prime Number Theorem for  $F(s)$  concerns the asymptotic behavior of the counting function

$$\psi_F(x) = \sum_{n \leq x} \Lambda(n) b_F(n) = \sum_{n \leq x} \Lambda_F(n).$$

It is expected that  $\psi_F(x) = m_F x + o(x)$ . In [8] Kaczorowski and Perelli shows that the Prime Number Theorem is equivalent to the non-vanishing on the 1-line. The proof is based on a weak zero-density estimate near the 1-line and on a

simple almost periodicity argument. Furthermore, if  $F$  a function non-vanishing at  $Re(s) = 1$ , then a direct application of the Perron formula [27] to Dirichlet series  $-\frac{F'}{F}$  implies that for any  $x > 1$  (not an integer) and  $T > 1$  one has

$$\psi_F(x) = m_F x - \sum_{|Im(\rho)| \leq T} \frac{x^\rho}{\rho} + O\left(\frac{x^{1+\epsilon}}{T}\right),$$

for some small  $\epsilon > 0$ . We let  $T \rightarrow \infty$  and multiply the above formula with  $\frac{\log^l x}{x}$ . Smajlovic [26] proved the following equivalence.

$$\forall l \in \mathbb{N}, \lim_{x \rightarrow \infty} \log^l x \left( m_F - \frac{\psi_F(x)}{x} \right) = 0 \iff \lim_{x \rightarrow \infty} \log^l x \sum_{\rho} \frac{x^{1-\rho}}{\rho} = 0. \quad (5)$$

Denote  $\tilde{\mathcal{S}}$  the set of all functions  $F \in \mathcal{S}$ , non-vanishing on the line  $Re(s) = 1$  and such that (5) holds true for all positive integers  $l$ . Obviously  $\tilde{\mathcal{S}} \subset \mathcal{S}$ . From (5), we have

$$F \in \tilde{\mathcal{S}} \iff F \in \mathcal{S} \text{ and } \psi_F(x) = m_F x + o\left(\frac{x}{\log^l x}\right)$$

and if a function  $F \in \mathcal{S}$  has a Landau type zero free region (similar to  $\mathcal{H}$  above), then  $F \in \tilde{\mathcal{S}}$ . To prove the second assertion, note that the Landau type zero free region implies, by standard analytic arguments that the error term in the prime number theorem is  $O(\exp(-c\sqrt{\log x}))$ , hence (5) holds true. **Finally, let us note here that Theorem 2 not hold true only for functions  $F \in \mathcal{S}$  having a Landau type zero free region but valid for much larger class  $\tilde{\mathcal{S}}$ .** Therefore, in Section 3, we will consider Smajlovic class  $\tilde{\mathcal{S}}$  of  $L$ -functions for our study on the modified Li criterion and the modified Li coefficients.

An asymptotic formula of the number  $\lambda_F(n)$  was proved in [18] inspired from Lagarias method [9] yields to a sharper error term  $O(\sqrt{n} \log n)$ . To do so, we use the arithmetic formula (4). Furthermore, we prove that is equivalent to the Riemann hypothesis.

**Theorem 3.** *Let  $F \in \mathcal{S}$ . Then*

$$RH \iff \lambda_F(n) = \frac{d_F}{2} n \log n + c_F n + O(\sqrt{n} \log n),$$

where

$$c_F = \frac{d_F}{2}(\gamma - 1) + \frac{1}{2} \log(\lambda Q_F^2), \quad \lambda = \prod_{j=1}^r \lambda_j^{2\lambda_j}$$

and  $\gamma$  is the Euler constant.

Theorem 3 was proved also in [11] using the saddle-point method in conjunction with the theory of the Nörlund-Rice integrals.

### 3 Reformulation of the Li criterion and Modified Li's coefficients

Let  $F \in \tilde{\mathcal{S}}$ . Define the "modified Li-coefficients" as follows

$$\lambda_F(n, a) = \sum_{\rho \in Z(F)} \left[ 1 - \left( \frac{\rho - a}{\rho + a - 1} \right)^n \right],$$

where the sum over  $\rho$  is

$$\sum_{\rho} = \lim_{T \rightarrow \infty} \sum_{|\operatorname{Im} \rho| \leq T}.$$

The last sum is  $*$ -convergent for all  $n \in \mathbb{N}$  and

$$\operatorname{Re}(\lambda_F(n, a)) = \sum_{\rho \in Z(F)} \operatorname{Re} \left[ 1 - \left( \frac{\rho - a}{\rho + a - 1} \right)^n \right]$$

converges absolutely for all  $n \in \mathbb{N}$ . The multi-set  $\mathcal{Z}$  of zeros of  $\xi_F(s)$  is invariant under the map  $\rho \mapsto 1 - \bar{\rho}$  implies that  $\lambda_F(-n, a) = \overline{\lambda_F(n, a)}$ . Then  $\operatorname{Re} \lambda_F(-n, a) = \operatorname{Re} \lambda_F(n, a)$  for all  $n \in \mathbb{N}$ . To obtain a new reformulation of the Li criterion we need to modify Bombieri-Lagarias Theorem [3, Theorem 1].

**Lemma 1. (Modified Bombieri-Lagarias Theorem).** *Let  $\beta$  and  $a$  be a real numbers such that  $\beta > a$  and  $R$  a multiset of complex numbers  $\rho$  such that :*

- i)  $a - 2\beta \notin R$ ,*
- ii)*

$$\sum_{\rho \in R} \frac{1 + |\operatorname{Re}(\rho)|}{(1 + |\rho + a - 2\beta|)^2} < +\infty.$$

*Then, the following conditions are equivalent*

- 1.  $\operatorname{Re}(\rho) < \beta$  for all  $\rho$ .*
- 2.  $\sum_{\rho \in R} \operatorname{Re} \left[ 1 - \left( \frac{\rho - a}{\rho + a - 1} \right)^n \right] \geq 0$ , for all  $n = 1, 2, \dots$*

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*If we take  $\beta < a$ , we just change 2) and 3) as follows :*

- 2)  $\sum_{\rho \in R} \operatorname{Re} \left[ 1 - \left( \frac{\rho - a}{\rho + a - 1} \right)^n \right] \leq 0$ , for all  $n = 1, 2, \dots$*
- 3) For every fixed  $\epsilon > 0$ , there is a constant  $c(\epsilon) > 0$  such that*

$$\sum_{\rho \in R} \operatorname{Re} \left[ 1 - \left( \frac{\rho - a}{\rho + a - 1} \right)^n \right] \leq c(\epsilon) e^{\epsilon n}, \text{ for all } n = 1, 2, \dots$$



3. For every fixed  $\epsilon > 0$ , there is a constant  $c(\epsilon) > 0$  such that

$$\sum_{\rho \in R} \operatorname{Re} \left[ 1 - \left( \frac{\rho - a}{\rho + a - 1} \right)^n \right] \geq -c(\epsilon) e^{\epsilon n}, \text{ for all } n = 1, 2, \dots$$

**Remark.** If  $\rho$  and  $\bar{\rho}$  are in  $R$ , we don't need to take the real part since the expression is real.

*Proof.* For the proof it suffices to observe that for  $\rho = \beta + i\gamma$ , one has

$$\left| \frac{\rho - a}{\rho + a - 2\beta} \right| = \left| \frac{\beta - a + i\gamma}{a - \beta + i\gamma} \right| = 1$$

and for  $\rho = \beta' + i\gamma$

$$\left| \frac{\rho - a}{\rho + a - 2\beta} \right|^2 = 1 + \frac{4(\beta - a)(\beta' - \beta)}{|\rho + a - 2\beta|^2}.$$

Then, Lemma 1 follows by adapting Bombieri and Lagarias proof [3, Theorem 1].  $\square$

Using Lemma 1, we deduce a new reformulation of the Li criterion (or the modified Li's criterion).

**Theorem 4.** Let  $\beta$  and  $a$  be a real numbers such that  $\beta > a$  and  $R$  a multiset of complex numbers  $\rho$  such that :

- i)  $a - 2\beta \notin R$ ,  $-a \notin R$ ,
- ii)

$$\sum_{\rho \in R} \frac{1 + |\operatorname{Re}(\rho)|}{(1 + |\rho + a - 2\beta|)^2} < +\infty$$

and

$$\sum_{\rho \in R} \frac{1 + |\operatorname{Re}(\rho)|}{(1 + |\rho - a|)^2} < +\infty.$$

- iii) If  $\rho \in R$  then  $2\beta - \rho \in R$  with the same multiplicity as  $\rho$ .
- Then, the following conditions are equivalent

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If we take  $\beta < a$ , we just change 2) and 3) as follows :

- 2)  $\sum_{\rho \in R} \operatorname{Re} \left[ 1 - \left( \frac{\rho - a}{\rho + a - 1} \right)^n \right] \leq 0$ , for all  $n = 1, 2, \dots$
- 3) For every fixed  $\epsilon > 0$ , there is a constant  $c(\epsilon) > 0$  such that

$$\sum_{\rho \in R} \operatorname{Re} \left[ 1 - \left( \frac{\rho - a}{\rho + a - 1} \right)^n \right] \leq c(\epsilon) e^{\epsilon n}, \text{ for all } n = 1, 2, \dots$$

1.  $Re(\rho) = \beta$  for all  $\rho$ .
2.  $\sum_{\rho \in R} Re \left[ 1 - \left( \frac{\rho - a}{\rho + a - 1} \right)^n \right] \geq 0$ , for all  $n = 1, 2, \dots$
3. For every fixed  $\epsilon > 0$ , there is a constant  $c(\epsilon) > 0$  such that

$$\sum_{\rho \in R} Re \left[ 1 - \left( \frac{\rho - a}{\rho + a - 1} \right)^n \right] \geq -c(\epsilon) e^{\epsilon n}, \text{ for all } n = 1, 2, \dots$$

*Proof.* The proof is the same as in [3, Theorem 1], so we omit it.  $\square$

Now, we are ready to state a reformulation of the Li criterion for the Selberg class.

**Theorem 5.** *Let  $a$  be a real number such that  $a < 1/2$  and  $F \in \tilde{\mathcal{S}}$  be a function such that  $a \notin Z(F)$ . Then, all non-trivial zeros of  $F$  lie on the line  $Re(s) = 1/2$  if and only if  $Re(\lambda_F(n, a)) \geq 0$  for all  $n \in \mathbb{N}$ .*

*Proof.* Theorem 4 with  $\beta = 1/2$  and  $R$  is the multiset  $Z(F)$  yields to  $Re(\rho) = 1/2$  if and only if  $Re(\lambda_F(n, a)) \geq 0$  for all  $n \in \mathbb{N}$ .  $\square$

#### Remarks.

- From Theorem 5 and the footnote 3, we don't need to call this criterion as in the origin definition of Li [10] for the classical zeta function "positivity Li criterion".
- Theorem 5 can be proved without the use of Theorem 4. Indeed, let  $a < 1/2$  and  $\rho = \beta + i\gamma$  a non-trivial zero of  $F$ . Observe that

$$\left| \frac{\rho - a}{\rho + a - 1} \right|^2 = 1 + \frac{(1 - 2a)(2\beta - 1)}{|\rho + a - 1|^2}.$$

Therefore, there exist at least  $\rho$  such that  $\left| \frac{\rho - a}{\rho + a - 1} \right| > 1$ . Since  $\frac{(1-2a)(2\beta-1)}{|\rho+a-1|^2}$  tends to 0 if  $|\rho|$  tends to  $\infty$ . Then, the maximum over  $\rho$  is achieved and the only finitely zeros  $\rho_k$  such that  $\left| \frac{\rho - a}{\rho + a - 1} \right| = 1 + t = k = \max$ . For the remainder zeros  $\left| \frac{\rho - a}{\rho + a - 1} \right| \leq 1 + t - \delta$  for some  $\delta > 0$ . Hence

$$1 - \left( \frac{\rho_k - a}{\rho_k + a - 1} \right)^n = 1 - (1 + t)^n e^{in\theta_k},$$

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If we assume  $a > 1/2$ , then the modified Li criterion is written as follows :  
All non-trivial zeros of  $F$  lie on the line  $Re(s) = 1/2$  if and only if  $Re(\lambda_F(n, a)) \leq 0$  for all  $n \in \mathbb{N}$ .

for  $n$  large. Then, using Dirichlet's theorem, we obtain

$$\begin{aligned} \sum_{\rho} 1 - \left( \frac{\rho - a}{\rho + a - 1} \right)^n &= \sum_{\rho_k} 1 - \left( \frac{\rho_k - a}{\rho_k + a - 1} \right)^n + \sum_{\rho \neq \rho_k} 1 - \left( \frac{\rho_k - a}{\rho_k + a - 1} \right)^n \\ &= K(1 - (1 + t)^n) + O(n^2(1 + t - \delta)^n). \end{aligned}$$

- Using that

$$\xi_F(s) = \xi_F(0) \prod_{\rho \in Z(F)} \left( 1 - \frac{s}{\rho} \right),$$

we can deduce easily that

$$\begin{aligned} \lambda_F(n, a) &= \sum_{\rho \in Z(F)} \left[ 1 - \left( \frac{\rho - a}{\rho + a - 1} \right)^n \right] \\ &= \frac{1}{(n-1)!} \frac{d^n}{ds^n} \left[ (s-a)^{n-1} \log \xi_F(s) \right]_{s=1-a}. \end{aligned} \quad (6)$$

Then, we have

$$\frac{d}{ds} \log \xi_F \left( \frac{s-a}{s+a-1} \right) = \sum_{n=0}^{\infty} \lambda_F(-(n+1), a) (s-a)^n. \quad (7)$$

As a consequence, we can prove in another way the modified Li criterion on the class  $\tilde{\mathcal{S}}$  by the same argument as in [4, Theorem 1]

## 4 Arithmetic formula for the modified Li coefficients

$\lambda_F(n, a)$

In this section, we obtain an arithmetic formula for  $\lambda_F(n, a)$ . The proof is based on the use of the Weil explicit formula written in the context of the Selberg class

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Another method to prove equation (6) is to consider the function

$$h(z) = -\frac{n(2a-1)(s-a)^{n-1}}{(s+a-1)^{n+1}} + \frac{n(2a-1)}{(s+a-1)^2}$$

and apply the Littlewood Theorem to the integral

$$\int_C h(s) \log \xi_F(s) ds$$

with  $C$  is a rectangular contour with vertices at  $\pm T \pm iT$  with real  $T$  tends to infity and not coincid with any zero of  $\xi_F(s)$ .

with a suitable test function.

First, let recall the Weil explicit formula.

**Proposition 1.** [15, 16] *Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  satisfy the following conditions:*

- *$f$  is normalized,*

$$2f(x) = f(x+0) + f(x-0), \quad x \in \mathbb{R}.$$

- *There is a number  $b > 0$  such that*

$$V_{\mathbb{R}} \left( f(x) e^{\left(\frac{1}{2}+b\right)|x|} \right) < \infty,$$

*where  $V_{\mathbb{R}}(\cdot)$  denotes the total variation on  $\mathbb{R}$ .*

- *For all  $1 \leq j \leq r$ , the function  $G_j(x) = f(x) e^{-ix \frac{Im(\mu_j)}{\lambda_j}}$  satisfies*

$$G_j(x) + G_j(-x) = 2f(0) + O(|x|^\epsilon), \quad \epsilon > 0.$$

*Let  $F(s) \in \mathcal{S}$ . Then,*

$$\begin{aligned} \sum_{\rho} H(\rho) &= m_F (H(0) + H(1)) + 2f(0) \log Q_F \\ &+ \sum_{j=1}^r \int_0^{+\infty} \left\{ \frac{2\lambda_j G_j(0)}{x} - \frac{e^{\left[\left(1-\frac{\lambda_j}{2} - Re(\mu_j)\right) \frac{x}{\lambda_j}\right]}}{1 - e^{-\frac{x}{\lambda_j}}} (G_j(x) + G_j(-x)) \right\} e^{-\frac{x}{\lambda_j}} dx \\ &- \sum_{n=1}^{\infty} \left[ \frac{\Lambda_F(n)}{\sqrt{n}} f(\log n) + \frac{\overline{\Lambda_F(n)}}{\sqrt{n}} f(-\log n) \right], \end{aligned} \quad (8)$$

*where*

$$H(s) = \int_{-\infty}^{+\infty} f(x) e^{(s-1/2)x} dx \quad \text{et} \quad \sum_{\rho} H(\rho) = \lim_{T \rightarrow \infty} \sum_{|Im(\rho)| < T} H(\rho).$$

For more details about the proof of the above proposition, see for example the paper [2] of Barner. Using [2, Proposition page.146] the integral in the sum in the third term in the right-hand side of (3) can be written as follows

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\widehat{G}_j(x) + \widehat{G}_j(-x)}{2} \psi \left( \frac{\lambda_j}{2} + Re(\mu_j) + i\lambda_j x \right) dx,$$

where

$$\widehat{G_j}(x) = \int_{-\infty}^{+\infty} G_j(t) e^{itx} dt$$

is the Fourier transform of  $G_j$  and  $\psi(s) = \frac{\Gamma'}{\Gamma}(s)$ .

Similar to [3, Lemma 2], we have the following result.

**Lemma 2.** *Let  $a$  be a real number. For  $n = 1, 2, \dots$  let consider the function*

$$f_n(x) = \begin{cases} P_n(x) & \text{if } -\infty < x < 0, \\ \frac{n}{2}(1-2a) & \text{if } x = 0, \\ 0 & \text{if } 0 < x, \end{cases}$$

where

$$P_n(x) = e^{(1/2-a)x} \sum_{l=1}^n \binom{n}{l} \frac{(1-2a)^l}{(l-1)!} x^{l-1}.$$

Then

$$H_n(s) = \int_{-\infty}^{+\infty} f_n(x) e^{(s-1/2)x} dx = 1 - \left(1 + \frac{2a-1}{s-a}\right)^n.$$

Note that, for  $a = 0$ , we have  $\lambda_F(-n, a) = \lambda_F(-n)$  which its arithmetic formula were stated in Theorem 2.

**Theorem 6.** Let  $F \in \tilde{\mathcal{S}}$ . For  $a < 0$ , we have

$$\begin{aligned}
& \lambda_F(-n, a) \\
&= m_F \left[ 2 - \left( 1 - \frac{2a-1}{a} \right)^n - \left( 1 + \frac{2a-1}{1-a} \right)^n \right] + n(1-2a) \left[ \log Q_F - \frac{d_F}{2} \gamma \right] \\
&\quad - \sum_{l=1}^n \binom{n}{l} \frac{(2a-1)^l}{(l-1)!} \lim_{X \rightarrow +\infty} \left\{ \sum_{k \leq X} \frac{\Lambda_F(k)}{k^{1-a}} \log^{l-1} k - \frac{m_F(l-1)!}{X^{-a}} \sum_{k=0}^{l-1} \frac{\log^k X}{k!(-a)^{l-k}} \right\} \\
&\quad + n(1-2a) \sum_{j=1}^r \lambda_j \left( -\frac{1}{\lambda_j + \mu_j} + \sum_{l=1}^{\infty} \frac{\lambda_j + \mu_j}{l(l + \lambda_j + \mu_j)} \right) \\
&\quad - n(1-2a) \sum_{j=1}^r \lambda_j \sum_{k=0}^{+\infty} (a\lambda_j)^k \sum_{m=0}^{+\infty} \frac{1}{(m + \lambda_j + \mu_j)^{k+1}} \\
&\quad + \sum_{j=1}^r \sum_{l=2}^n \binom{n}{l} (-(1-2a)\lambda_j)^l \sum_{m=0}^{+\infty} \frac{1}{(m + (1-a)\lambda_j + \mu_j)^l}.
\end{aligned} \tag{9}$$

*Proof.* For large  $X$  not integer, let

$$f_{n,X}(x) = \begin{cases} f_n(x) & \text{if } -\log X < x < 0, \\ \frac{1}{2}f_n(-\log X) & \text{if } x = -\log X, \\ \frac{n}{2}(1-2a) & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the function  $f_{n,X}(x)$  satisfies condition of Proposition 1. Let

$$H_{n,X}(s) = \int_{-\infty}^{+\infty} f_{n,X}(x) e^{(s-1/2)x} dx.$$

Therefore, we get

$$W_{\lambda_j, \mu_j} = \sum_{j=1}^r \int_0^{+\infty} \left\{ \frac{2\lambda_j G_{j,n,X}(0)}{x} - \frac{e^{\left[ \left( 1 - \frac{\lambda_j}{2} - \operatorname{Re}(\mu_j) \right) \frac{x}{\lambda_j} \right]}}{1 - e^{-\frac{x}{\lambda_j}}} (G_{j,n,X}(x) + G_{j,n,X}(-x)) \right\} e^{-\frac{x}{\lambda_j}} dx,$$

with

$$G_{j,n,X}(x) = f_{n,X}(x) e^{-ix \frac{\operatorname{Im}(\mu_j)}{\lambda_j}}.$$

---

When  $a = 0$ , we get  $\lambda_F(-n, 0) = \lambda_F(-n)$  as given in [16, Theorem 2] or [26]. If we replace  $-a$  by  $b$ , we can assumed  $b > 0$  and this finds Sekatskii's arithmetic formula of the modified coefficients  $\lambda_\zeta(n, b)$  (see. [24]).

Using the relation

$$\frac{\Gamma'}{\Gamma}(z) = \int_0^{+\infty} \left( \frac{e^{-u}}{u} - \frac{e^{-zu}}{1 - e^{-u}} \right) du,$$

we obtain

$$\begin{aligned}
& \int_0^{+\infty} \left\{ \frac{2\lambda_j G_{j,n,X}(0)}{x} - \frac{e^{\left[\left(1-\frac{\lambda_j}{2}-\operatorname{Re}(\mu_j)\right)\frac{x}{\lambda_j}\right]}}{1-e^{-\frac{x}{\lambda_j}}} (G_{j,n,X}(x) + G_{j,n,X}(-x)) \right\} e^{-\frac{x}{\lambda_j}} dx \\
&= \lim_{X \rightarrow +\infty} \int_0^{\log X} \left\{ \frac{2\lambda_j f_{n,X}(0)}{x} - \frac{e^{\left[\left(1-\frac{\lambda_j}{2}-\operatorname{Re}(\mu_j)\right)\frac{x}{\lambda_j}\right]}}{1-e^{-\frac{x}{\lambda_j}}} f_{n,X}(-x) e^{ix \frac{\operatorname{Im}(\mu_j)}{\lambda_j}} \right\} e^{-\frac{x}{\lambda_j}} dx \\
&= \lim_{X \rightarrow +\infty} \int_0^{\log X} \left\{ \frac{n(1-2a)\lambda_j}{x} - \frac{e^{\left[\left(1-\frac{\lambda_j}{2}-\overline{\mu_j}\right)\frac{x}{\lambda_j}\right]}}{1-e^{-\frac{x}{\lambda_j}}} f_{n,X}(-x) \right\} e^{-\frac{x}{\lambda_j}} dx \\
&= \lim_{X \rightarrow +\infty} \lambda_j \int_0^{\log X} \left\{ \frac{n(1-2a)e^{-x}}{x} - \frac{e^{-\left(\frac{\lambda_j}{2}+\overline{\mu_j}\right)x}}{1-e^{-x}} f_{n,X}(-\lambda_j x) \right\} dx \\
&= n\lambda_j(1-2a) \frac{\Gamma'}{\Gamma}(\lambda_j + \overline{\mu_j}) + \lambda_j \lim_{X \rightarrow +\infty} \int_0^{\frac{\log X}{\lambda_j}} \left\{ \frac{n(1-2a)e^{-(\lambda_j+\overline{\mu_j})x}}{1-e^{-x}} - \frac{e^{-\left(\frac{\lambda_j}{2}+\overline{\mu_j}\right)x}}{1-e^{-x}} f_n(-\lambda_j x) \right\} dx \\
&= n\lambda_j(1-2a) \frac{\Gamma'}{\Gamma}(\lambda_j + \overline{\mu_j}) \\
&\quad + \lambda_j \int_0^{+\infty} \left\{ \frac{n(1-2a)e^{-(\lambda_j+\overline{\mu_j})x}}{1-e^{-x}} - \frac{e^{-\left(\frac{\lambda_j}{2}+\overline{\mu_j}\right)x}}{1-e^{-x}} \left[ e^{-(\frac{1}{2}-a)\lambda_j x} \sum_{l=1}^n \binom{n}{l} \frac{(1-2a)^l}{(l-1)!} (-\lambda_j x)^{l-1} \right] \right\} dx \\
&= n\lambda_j(1-2a) \frac{\Gamma'}{\Gamma}(\lambda_j + \overline{\mu_j}) \\
&\quad + \lambda_j \int_0^{+\infty} \left\{ n(1-2a) - e^{a\lambda_j x} \sum_{l=1}^n \binom{n}{l} \frac{(1-2a)^l}{(l-1)!} (-\lambda_j x)^{l-1} \right\} \frac{e^{-(\lambda_j+\overline{\mu_j})x}}{1-e^{-x}} dx \\
&= n\lambda_j(1-2a) \frac{\Gamma'}{\Gamma}(\lambda_j + \overline{\mu_j}) \\
&\quad + \lambda_j \int_0^{+\infty} \left\{ n(1-2a)(1-e^{a\lambda_j x}) - e^{a\lambda_j x} \sum_{l=2}^n \binom{n}{l} \frac{(1-2a)^l}{(l-1)!} (-\lambda_j x)^{l-1} \right\} \frac{e^{-(\lambda_j+\overline{\mu_j})x}}{1-e^{-x}} dx \\
&= n\lambda_j(1-2a) \frac{\Gamma'}{\Gamma}(\lambda_j + \overline{\mu_j}) + \lambda_j \int_0^{+\infty} \left\{ n(1-2a)(1-e^{a\lambda_j x}) \frac{e^{-(\lambda_j+\overline{\mu_j})x}}{1-e^{-x}} \right\} dx \\
&\quad + \lambda_j(1-2a) \int_0^{+\infty} \left\{ -\sum_{l=2}^n \binom{n}{l} \frac{(-(1-2a)\lambda_j x)^{l-1}}{(l-1)!} \right\} \frac{e^{-((1-a)\lambda_j+\overline{\mu_j})x}}{1-e^{-x}} dx \\
&= n\lambda_j(1-2a) \frac{\Gamma'}{\Gamma}(\lambda_j + \overline{\mu_j}) + n(1-2a)\lambda_j I_1 + (1-2a)\lambda_j I_2,
\end{aligned}$$



where

$$I_1 = \int_0^{+\infty} \left\{ (1 - e^{a\lambda_j x}) \frac{e^{-(\lambda_j + \overline{\mu_j})x}}{1 - e^{-x}} \right\} dx$$

and

$$I_2 = \int_0^{+\infty} \left\{ - \sum_{l=2}^n \binom{n}{l} \frac{(-(1-2a)\lambda_j x)^{l-1}}{(l-1)!} \right\} \frac{e^{-((1-a)\lambda_j + \overline{\mu_j})x}}{1 - e^{-x}} dx.$$

In order that the integral  $I_1$  and  $I_2$  converges, one needs to assume that  $\operatorname{Re}((a-1)\lambda_j - \mu_j) < 0$ . Since the assumption on  $\lambda_j$  and  $\mu_j$  posed in the second axiom in the definition of the Selberg class yield that, this is true only if  $a \leq 1$ .

Let start with  $I_2$ . For  $a \leq 1$ , we have

$$I_2 = \sum_{l=2}^n \binom{n}{l} (-(1-2a)\lambda_j)^{l-1} \sum_{m=0}^{+\infty} \frac{1}{(m + (1-a)\lambda_j + \overline{\mu_j})^l}. \quad (11)$$

Furthermore, for  $a \leq 1$ , one has,

$$\begin{aligned} I_1 &= \frac{\Gamma'}{\Gamma}(\lambda_j(1-a) + \overline{\mu_j}) - \frac{\Gamma'}{\Gamma}(\lambda_j + \overline{\mu_j}) \\ &= - \sum_{k=1}^{+\infty} \frac{(a\lambda_j)^k}{k!} \int_0^{+\infty} x^k \frac{e^{-(\lambda_j + \overline{\mu_j})x}}{1 - e^{-x}} dx \\ &= - \sum_{k=1}^{+\infty} (a\lambda_j)^k \sum_{m=0}^{+\infty} \frac{1}{(m + \lambda_j + \overline{\mu_j})^{k+1}}. \end{aligned} \quad (12)$$

In the other hand, a simple computation yields

$$H_{n,X}(0) = 1 - \left(1 - \frac{2a-1}{a}\right)^n + \sum_{l=1}^n \binom{n}{l} (2a-1)^l X^a \sum_{k=0}^{l-1} \frac{\log^k X}{k!(-a)^{l-k}}, \quad (13)$$

$$\begin{aligned} H_{n,X}(1) &= 1 - \left(1 + \frac{2a-1}{1-a}\right)^n + \sum_{l=1}^n \binom{n}{l} (2a-1)^l X^{a-1} \sum_{k=0}^{l-1} \frac{\log^k X}{k!(1-a)^{l-k}} \\ &= 1 - \left(1 + \frac{2a-1}{1-a}\right)^n + O\left(\frac{\log^n X}{X^{1-a}}\right) \end{aligned} \quad (14)$$

and

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left[ \frac{\Lambda_F(n)}{\sqrt{n}} f_{n,X}(\log n) + \frac{\overline{\Lambda_F(n)}}{\sqrt{n}} f_{n,X}(-\log n) \right] \\
&= \sum_{l=1}^n \binom{n}{l} \frac{(-1)^{l-1}}{(l-1)!} (1-2a)^l \sum_{k \leq X} \frac{\overline{\Lambda_F(k)}}{k^{1-a}} \log^{l-1} k. \tag{15}
\end{aligned}$$

Inserting equations (11) and (12) into formula (10), using equations (13), (14) and (15) and the explicit formula given by equation (8), for  $a < 0$ , we get

$$\begin{aligned}
& \lim_{X \rightarrow +\infty} \sum_{\rho} H_{n,X}(\rho) \\
&= m_F \left[ 2 - \left( 1 - \frac{2a-1}{a} \right)^n - \left( 1 + \frac{2a-1}{1-a} \right)^n \right] + n(1-2a) \left[ \log Q_F - \frac{d_F}{2} \gamma \right] \\
&\quad - \sum_{l=1}^n \binom{n}{l} \frac{(2a-1)^l}{(l-1)!} \lim_{X \rightarrow +\infty} \left\{ \sum_{k \leq X} \frac{\overline{\Lambda_F(k)}}{k^{1-a}} \log^{l-1} k - m_F(l-1)! X^a \sum_{k=0}^{l-1} \frac{\log^k X}{k!(-a)^{l-k}} \right\} \\
&\quad + n(1-2a) \sum_{j=1}^r \lambda_j \left( -\frac{1}{\lambda_j + \overline{\mu_j}} + \sum_{l=1}^{\infty} \frac{\lambda_j + \overline{\mu_j}}{l(l + \lambda_j + \overline{\mu_j})} \right) \\
&\quad - n(1-2a) \sum_{j=1}^r \lambda_j \sum_{k=0}^{+\infty} (a\lambda_j)^k \sum_{m=0}^{+\infty} \frac{1}{(m + \lambda_j + \overline{\mu_j})^{k+1}} \\
&\quad + \sum_{j=1}^r \sum_{l=2}^n \binom{n}{l} (-(1-2a)\lambda_j)^l \sum_{m=0}^{+\infty} \frac{1}{(m + (1-a)\lambda_j + \overline{\mu_j})^l}. \tag{16}
\end{aligned}$$

Under condition made above on  $F(s)$ , since  $a < 0$ , we prove by the same argument as in [15, page. 56 and 57] that

$$\lim_{X \rightarrow +\infty} \sum_{\rho} H_{n,X}(\rho) = \sum_{\rho} H(\rho) = \lambda_F(n, a).$$

Furthermore, recall that  $\lambda_F(-n, a) = \overline{\lambda_F(n, a)}$ , then, for  $a < 0$ , we obtain

$$\begin{aligned}
\lambda_F(-n, a) = & m_F \left[ 2 - \left( 1 - \frac{2a-1}{a} \right)^n - \left( 1 + \frac{2a-1}{1-a} \right)^n \right] + n(1-2a) \left[ \log Q_F - \frac{d_F}{2} \gamma \right] \\
& - \sum_{l=1}^n \binom{n}{l} \frac{(2a-1)^l}{(l-1)!} \lim_{X \rightarrow +\infty} \left\{ \sum_{k \leq X} \frac{\Lambda_F(k)}{k^{1-a}} \log^{l-1} k - m_F(l-1)! X^a \sum_{k=0}^{l-1} \frac{\log^k X}{k!(-a)^{l-k}} \right\} \\
& + n(1-2a) \sum_{j=1}^r \lambda_j \left( -\frac{1}{\lambda_j + \mu_j} + \sum_{l=1}^{\infty} \frac{\lambda_j + \mu_j}{l(l + \lambda_j + \mu_j)} \right) \\
& - n(1-2a) \sum_{j=1}^r \lambda_j \sum_{k=0}^{+\infty} (a\lambda_j)^k \sum_{m=0}^{+\infty} \frac{1}{(m + \lambda_j + \mu_j)^{k+1}} \\
& + \sum_{j=1}^r \sum_{l=2}^n \binom{n}{l} (-(1-2a)\lambda_j)^l \sum_{m=0}^{+\infty} \frac{1}{(m + (1-a)\lambda_j + \mu_j)^l}.
\end{aligned} \tag{17}$$

**Remark.** The arithmetic formula given in Theorem 6 can be written in a simple way as follows

$$\begin{aligned}
\lambda_F(-n, a) = & m_F \left[ 2 - \left( 1 - \frac{2a-1}{a} \right)^n - \left( 1 + \frac{2a-1}{1-a} \right)^n \right] + n(1-2a) \log Q_F \\
& - \sum_{l=1}^n \binom{n}{l} \frac{(2a-1)^l}{(l-1)!} \lim_{X \rightarrow +\infty} \left\{ \sum_{k \leq X} \frac{\Lambda_F(k)}{k^{1-a}} \log^{l-1} k - \frac{m_F(l-1)!}{X^{-a}} \sum_{k=0}^{l-1} \frac{\log^k X}{k!(-a)^{l-k}} \right\} \\
& + n(1-2a) \sum_{j=1}^r \psi(\lambda_j(1-a) + \mu_j) \\
& + \sum_{j=1}^r \sum_{l=2}^n \binom{n}{l} (-(1-2a)\lambda_j)^l \sum_{m=0}^{+\infty} \frac{1}{(m + (1-a)\lambda_j + \mu_j)^l}.
\end{aligned}$$

□

## 5 An asymptotic formula

In this section, we use the same argument as in [11, Theorem 4.1] which use the saddle point method in conjunction with the Nörlund-Rice integral to deduce an asymptotic formula for the modified Li coefficients equivalent to the Riemann hypothesis.

**Theorem 7.** *Let  $F \in \tilde{\mathcal{S}}$ . For  $a < 0$ , we have*

$$RH$$

$$\Longleftrightarrow$$

$$\begin{aligned} \lambda_F(n, a) &= \left( \frac{1}{2} - a \right) d_F n \log n \\ &+ \left\{ \frac{d_F}{2} (\gamma - 1 - \log(1 - 2a)) + \frac{1}{2} \log(\lambda Q_F^2) + C_F(a) \right\} (1 - 2a)n \\ &+ O(\sqrt{n} \log n), \end{aligned}$$

where

$$C_F(a) = \sum_{j=1}^r \sum_{k=0}^{+\infty} (a\lambda_j)^k \zeta(k, \lambda_j + \mu_j), \quad \lambda = \prod_{j=1}^r \lambda_j^{2\lambda_j},$$

$\gamma$  is the Euler constant and  $\zeta(s, q)$  is the Hurwitz zeta function given by

$$\zeta(s, q) = \sum_{n=0}^{+\infty} \frac{1}{(n+q)^s}.$$

*Proof.* Without loss of generality, we assume that  $\mu_j$  is a real number. First, write the arithmetic formula of  $\lambda_F(-n, a)$  (equation (18)) as

$$\begin{aligned} \lambda_F(-n, a) &= m_F \left[ 2 - \left( 1 - \frac{2a-1}{a} \right)^n - \left( 1 + \frac{2a-1}{1-a} \right)^n \right] \\ &+ n(1-2a) \left[ \log Q_F - \frac{d_F}{2} \gamma \right] + S_F(n, a) + n(1-2a)C_F(a) + S_1 + S_2, \end{aligned} \tag{18}$$

where

$$\begin{aligned} S_F(n, a) &= - \sum_{l=1}^n \binom{n}{l} \frac{(2a-1)^l}{(l-1)!} \lim_{X \rightarrow +\infty} \left\{ \sum_{k \leq X} \frac{\Lambda_F(k)}{k^{1-a}} \log^{l-1} k - m_F(l-1)! X^a \sum_{k=0}^{l-1} \frac{\log^k X}{k!(-a)^{l-k}} \right\}, \\ C_F(a) &= - \sum_{j=1}^r \lambda_j \sum_{k=0}^{+\infty} (a\lambda_j)^k \sum_{m=0}^{+\infty} \frac{1}{(m + \lambda_j + \mu_j)^{k+1}}, \\ S_1 &= n(1-2a) \sum_{j=1}^r \lambda_j \left( -\frac{1}{\lambda_j + \mu_j} + \sum_{l=1}^{\infty} \frac{\lambda_j + \mu_j}{l(l + \lambda_j + \mu_j)} \right) \end{aligned}$$

and

$$S_2 = \sum_{j=1}^r \sum_{l=2}^n \binom{n}{l} (-(1-2a)\lambda_j)^l \sum_{m=0}^{+\infty} \frac{1}{(m + (1-a)\lambda_j + \mu_j)^l}.$$

Write the sum  $S_2$  as follows

$$S_2 = \sum_{j=1}^r I_j,$$

where

$$\begin{aligned} I_j &= \sum_{l=2}^n \binom{n}{l} (-(1-2a)\lambda_j)^l \sum_{m=0}^{+\infty} \frac{1}{(m + (1-a)\lambda_j + \mu_j)^l} \\ &= \sum_{l=2}^n \binom{n}{l} (-1)^l \frac{\zeta(l, (1-a)\lambda_j + \mu_j)}{\left[ ((1-2a)\lambda_j)^{-1} \right]^l}, \end{aligned}$$

where

$$\zeta(s, q) = \sum_{n=0}^{+\infty} \frac{1}{(n+q)^s}$$

is the Hurwitz zeta function. Using the same notation of  $H_n(m, k)$  [11, Equation 4.1], we get

$$I_j = H_n \left( \frac{1-a}{1-2a} + \frac{\mu_j}{(1-2a)\lambda_j}, \frac{1}{(1-2a)\lambda_j} \right).$$

Now applying [11, Proposition 4.3] with  $m = \frac{1-a}{1-2a} + \frac{\mu_j}{(1-2a)\lambda_j}$  and  $k = \frac{1}{(1-2a)\lambda_j}$ , we obtain

$$\begin{aligned} I_j &= \left[ (1-a)\lambda_j + \mu_j - \frac{1}{2} \right] \\ &\quad - n(1-2a)\lambda_j \{ \psi((1-a)\lambda_j + \mu_j) - \log((1-2a)\lambda_j) + 1 - h_{n-1} \} \\ &\quad + a_n \left( \frac{1-a}{1-2a} + \frac{\mu_j}{(1-2a)\lambda_j}, \frac{1}{(1-2a)\lambda_j} \right), \end{aligned}$$

where the  $a_n$  are exponentially small and  $a_n = O(1)$  (see. [11, Proposition 4.3] for an explicit expression of  $a_n$ ). Here,  $h_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$  is a harmonic number, and  $\psi(x)$  is the logarithm derivative of the Gamma function. Therefore

$$\begin{aligned} I_j &= \left[ (1-a)\lambda_j + \mu_j - \frac{1}{2} \right] \\ &\quad - n(1-2a)\lambda_j [\psi((1-a)\lambda_j + \mu_j) - \log((1-2a)\lambda_j) + 1 - h_{n-1}] + O(1). \end{aligned} \tag{19}$$

Summing (19) over  $j$ , we get

$$\begin{aligned} \sum_{j=1}^r I_j &= \sum_{j=1}^r \left[ (1-a)\lambda_j + \mu_j - \frac{1}{2} \right] \\ &\quad - n(1-2a) \sum_{j=1}^r \lambda_j [\psi((1-a)\lambda_j + \mu_j) - \log((1-2a)\lambda_j) + 1 - h_{n-1}] + O(1). \end{aligned} \quad (20)$$

Using the expression

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{l=1}^{+\infty} \frac{z}{l(l+z)},$$

where  $\gamma$  is the Euler constant and the estimate

$$h_n = \log n - \gamma + \frac{1}{2n} + O\left(\frac{1}{2n^2}\right),$$

we deduce from (18), (20) and for  $a < 0$

$$\left[ 2 - \left( 1 - \frac{2a-1}{a} \right)^n - \left( 1 + \frac{2a-1}{1-a} \right)^n \right] = O(1)$$

that

$$\begin{aligned} \lambda_F(-n, a) &= \left( \sum_{j=1}^r \lambda_j \right) n(1-2a) \log n \\ &\quad + n(1-2a) \left\{ \left( \sum_{j=1}^r \lambda_j \right) (\gamma - 1) + \log Q_F - \sum_{j=1}^r \lambda_j \log[(1-2a)\lambda_j] + C_F(a) \right\} \\ &\quad + S_F(n, a) + O(1). \end{aligned} \quad (21)$$

With the notation  $d_F = \sum_{j=1}^r \lambda_j$  and  $\lambda = \prod_{j=1}^r \lambda_j^{2\lambda_j}$ , we obtain

$$\begin{aligned} \lambda_F(-n, a) &= (1/2 - a)d_F n \log n \\ &\quad + n(1-2a) \left\{ \frac{d_F}{2} [\gamma - 1 - \log(1-2a)] + \frac{1}{2} \log(\lambda Q_F)^2 + C_F(a) \right\} \\ &\quad + S_F(n, a) + O(1), \end{aligned} \quad (22)$$

where

$$S_F(n, a) = - \sum_{l=1}^n \binom{n}{l} \frac{(2a-1)^l}{(l-1)!} \lim_{X \rightarrow +\infty} \left\{ \sum_{k \leq X} \frac{\Lambda_F(k)}{k^{1-a}} \log^{l-1} k - m_F(l-1)! X^a \sum_{k=0}^{l-1} \frac{\log^k X}{k!(-a)^{l-k}} \right\}.$$

A simple adaptation of the argument used in [11, Lemma 4.4] yields to that, if the Riemann hypothesis holds for  $F \in \tilde{S}$ , then

$$S_F(n, a) = O(\sqrt{n} \log n).$$

Now, collecting all results above and noting that  $\lambda_F(-n, a) = \overline{\lambda_F(n, a)}$ , we deduce that the Riemann hypothesis implies

$$\begin{aligned} \lambda_F(n, a) &= (1/2 - a)d_F n \log n \\ &\quad + n(1 - 2a) \left\{ \frac{d_F}{2} [\gamma - 1 - \log(1 - 2a)] + \frac{1}{2} \log(\lambda Q_F)^2 + C_F(a) \right\} \\ &\quad + O(\sqrt{n} \log n). \end{aligned}$$

Conversely, if

$$\begin{aligned} \lambda_F(n, a) &= (1/2 - a)d_F n \log n \\ &\quad + n(1 - 2a) \left\{ \frac{d_F}{2} [\gamma - 1 - \log(1 - 2a)] + \frac{1}{2} \log(\lambda Q_F)^2 + C_F(a) \right\} \\ &\quad + O(\sqrt{n} \log n). \end{aligned}$$

then,  $\lambda_F(n, a)$  grows polynomially in  $n$ . Therefore, if the Riemann hypothesis is false then from Theorem 4, some modified Li coefficients become exponentially large in  $n$  and negative and the asymptotic formula of  $\lambda_F(n, a)$  rules out.  $\square$

## 6 Another proofs of Theorems 6 and 7

The arithmetic formula proved in Theorem 6 can be obtained by another way by using that the coefficients

$$\lambda_F(n, a) = \sum_{\rho \in Z(F)} \left[ 1 - \left( \frac{\rho - a}{\rho + a - 1} \right)^n \right] = \frac{1}{(n-1)!} \frac{d^n}{ds^n} [(s-a)^{n-1} \log \xi_F(s)]_{s=1-a}.$$

Therefore, the arithmetic formula (18) can be deduced by the same argument used by Smajlovic in [26, Appendix A] or by the author in [17]. Next, we will separately investigate the asymptotic behavior of archimedean and non-archimedean

contribution in the arithmetic formula given in Theorem 6 or established in another way below. To do so, we use a recurrence relation for the digamma function [1, 6.4.6], for  $n \geq 0$

$$\psi^{(n)}(z+1) = \psi^{(n)}(z) + (-1)^n n! z^{-n-1}$$

and

$$\psi^{(n)}(z) = (-1)^{n+1} n! \zeta(n+1, z) \quad (23)$$

for  $z \neq 0, -1, -2, \dots$ . We have,

$$\begin{aligned} \lambda_F(n, a) &= \frac{1}{(n-1)!} \frac{d^n}{ds^n} \left[ (s-a)^{n-1} \log \xi_F(s) \right]_{s=1-a} \\ &= \sum_{k=1}^n \binom{n}{k} \frac{(1-2a)^k}{(k-1)!} \left[ \frac{d^k}{ds^k} \log \xi_F(s) \right]_{s=1-a} \end{aligned}$$

Using that

$$\frac{\xi'_F}{\xi_F}(s) = \frac{m_F}{s} + \frac{m_F}{s-1} + \log Q_F + \frac{F'}{F}(s) + \sum_{j=1}^r \frac{\Gamma'}{\Gamma}(\lambda_j s + \mu_j),$$

we get

$$\begin{aligned} \lambda_F(n, a) &= \sum_{k=1}^n \binom{n}{k} \frac{(1-2a)^k}{(k-1)!} \left[ \frac{d^{k-1}}{ds^{k-1}} \frac{F'}{F}(s) \right]_{s=1-a} + m_F \sum_{k=1}^n \binom{n}{k} \frac{(1-2a)^k}{(1-a)^k} \\ &\quad + \sum_{k=1}^n \binom{n}{k} \frac{(1-2a)^k}{(k-1)!} \delta_{k,1} \log Q_F - m_F \sum_{k=1}^n \binom{n}{k} \frac{(1-2a)^k}{(-a)^k} \\ &\quad + \sum_{k=1}^n \binom{n}{k} \frac{(1-2a)^k}{(k-1)!} \sum_{j=1}^r \lambda_j^k \psi^{(k-1)}(\lambda_j(1-a) + \mu_j) \\ &= m_F \left[ 2 - \left( 1 - \frac{2a-1}{a} \right)^n - \left( 1 + \frac{2a-1}{1-a} \right)^n \right] + n(1-2a) \log Q_F \\ &\quad + \sum_{k=1}^n \binom{n}{k} (1-2a)^k \eta_{k-1}(F, 1-a) \\ &\quad + \sum_{k=1}^n \binom{n}{k} \frac{(1-2a)^k}{(k-1)!} \sum_{j=1}^r \lambda_j^k \psi^{(k-1)}(\lambda_j(1-a) + \mu_j), \quad (24) \end{aligned}$$

where  $\frac{F'}{F}(s) = \sum_{l=-1}^{+\infty} \eta_{k-1}(s - (1-a))^l$  is the Laurent expansion at  $s = 1-a$  and  $\delta_{k,1} = 1$  if  $k = 1$  and 0 otherwise. In particular,  $\eta_{-1} = -m_F$  if  $F$  has a pole



at  $s = 1 - a$  and  $\eta_{-1} = 0$  otherwise. It's easy to see that, for  $a < 0$

$$\begin{aligned}\eta_{k-1} &= \frac{1}{(k-1)!} \left[ \frac{d^{k-1}}{ds^{k-1}} \left( \frac{m_F}{s-1+a} + \frac{F'}{F}(s) \right) \right]_{1-a} \\ &= -\frac{1}{(k-1)!} \lim_{X \rightarrow +\infty} \left\{ \sum_{k \leq X} \frac{\Lambda_F(k)}{k^{1-a}} \log^{l-1} k - m_F(l-1)! X^a \sum_{k=0}^{l-1} \frac{\log^k X}{k!(-a)^{l-k}} \right\}.\end{aligned}\tag{25}$$

Furthermore, the last sum in equation (24) can be written as follows

$$\begin{aligned}& \sum_{k=1}^n \binom{n}{k} \frac{(1-2a)^k}{(k-1)!} \sum_{j=1}^r \lambda_j^k \psi^{(k-1)}(\lambda_j(1-a) + \mu_j) \\ &= (1-2a)n \sum_{j=1}^r \lambda_j \psi(\lambda_j(1-a) + \mu_j) \\ & \quad + \sum_{k=2}^n \binom{n}{k} (1-2a)^k \sum_{j=1}^r (-\lambda_j)^k \sum_{m=0}^{+\infty} \frac{1}{(\lambda_j(1-a) + \mu_j + m)^k}.\end{aligned}\tag{26}$$

Then, from equations (24), (25), (26), we find the arithmetic formula given in Theorem 6.

Now, to prove the asymptotic formula given in Theorem 7, we write equation 24

$$\lambda_F(n, a) = S_{Arch} + S_{NArch},$$

where

$$S_{Arch} = n(1-2a) \log Q_F + \sum_{k=1}^n \binom{n}{k} \frac{(1-2a)^k}{(k-1)!} \sum_{j=1}^r \lambda_j^k \psi^{(k-1)}(\lambda_j(1-a) + \mu_j)$$

and

$$S_{NArch} = m_F \left[ 2 - \left( 1 - \frac{2a-1}{a} \right)^n - \left( 1 + \frac{2a-1}{1-a} \right)^n \right] + \sum_{k=1}^n \binom{n}{k} (1-2a)^k \eta_{k-1}(F, 1-a).$$

Formula (23) yields to

$$\begin{aligned}
& \sum_{k=1}^n \binom{n}{k} \frac{(1-2a)^k}{(k-1)!} \sum_{j=1}^r \lambda_j^k \psi^{(k-1)}(\lambda_j(1-a) + \mu_j) \\
&= \sum_{k=1}^n \binom{n}{k} \frac{(1-2a)^k}{(k-1)!} \sum_{j=1}^r \lambda_j^k \psi^{(k-1)}(\lambda_j(1-a) + \mu_j + 1) \\
&\quad + \sum_{k=1}^n \binom{n}{k} \frac{(1-2a)^k}{(k-1)!} \sum_{j=1}^r \lambda_j^k (-1)^{k-1} (k-1)! (\lambda_j(1-a) + \mu_j)^{-k} \\
&= (1-2a)n \sum_{j=1}^r \lambda_j \psi(\lambda_j(1-a) + \mu_j + 1) \\
&\quad + \sum_{k=2}^n \binom{n}{k} (-(1-2a)\lambda_j)^k \sum_{j=1}^r \zeta(k, \lambda_j(1-a) + \mu_j + 1) \\
&\quad + \sum_{j=1}^r \left( \frac{a\lambda_j + \mu_j}{\lambda_j(1-a) + \mu_j} \right)^n - r. \tag{27}
\end{aligned}$$

Then

$$\begin{aligned}
S_{Arch} &= n(1-2a) \log Q_F + (1-2a)n \sum_{j=1}^r \lambda_j \psi(\lambda_j(1-a) + \mu_j + 1) \\
&\quad + \sum_{k=2}^n \binom{n}{k} (-(1-2a)\lambda_j)^k \sum_{j=1}^r \zeta(k, \lambda_j(1-a) + \mu_j + 1) \\
&\quad + \sum_{j=1}^r \left( \frac{a\lambda_j + \mu_j}{\lambda_j(1-a) + \mu_j} \right)^n - r. \tag{28}
\end{aligned}$$

Let

$$\sum_{j=1}^r \sum_{k=2}^n \binom{n}{k} (-(1-2a)\lambda_j)^k \zeta(k, \lambda_j(1-a) + \mu_j + 1) = \sum_{j=1}^r S_{Arch}(n, r).$$

To estimate  $S_{Arch}(n, r)$ , we follow the proof of the analogous theorem proved in [21, Theorem 2]. Calculus of residues implies that

$$\begin{aligned}
S_{Arch}(n, r) &= \frac{(-1)^n}{2\pi i} n! \int_R \frac{\Gamma(s-n)}{\Gamma(s+1)} ((1-2a)\lambda_j)^k \zeta(k, \lambda_j(1-a) + \mu_j + 1) ds \\
&= \frac{(-1)^n}{2\pi i} n! \int_R J_r(s) ds, \tag{29}
\end{aligned}$$

where  $R$  is a positively oriented rectangle with vertices  $3/2 \pm i$  and  $n + 1/2 \pm i$  containing simple poles  $s = 2, 3, \dots, n$  of  $J_r(s)$ . Residues are easily found using the fact that  $Res_{s=-n}\Gamma(s) = \frac{(-1)^n}{n!}$ , hence, for  $k = 2, \dots, n$ , we get

$$Res_{s=k}J_r(s) = \frac{1}{k!} ((1-2a)\lambda_j)^k \zeta(k, \lambda_j(1-a) + \mu_j + 1) \frac{(-1)^{n-k}}{(n-k)!}.$$

This justify the equality (29). The function  $J_r(s)$  is uniformly bounded on the real segment joining  $n + 1/2$  and  $e^n$ , hence, the rectangle  $R$  can be deformed to the line  $(e^n - i\infty, e^n + i\infty)$ . Further singularities of the function  $J_r(s)$  are a simple pole at  $s = 0$  and a pole  $s = 1$  of order 2. Therefore, we get

$$\begin{aligned} S_{Arch}(n, r) &= (-1)^{n-1} n! (Res_{s=0}J_r(s) + Res_{s=1}J_r(s)) \\ &\quad + O\left(n! \int_{-\infty}^{+\infty} \frac{\Gamma(e^n + it - n)}{\Gamma(e^n + it + 1)} ((1-2a)\lambda_j)^{e^n} \zeta(e^n, \lambda_j(1-a) + \mu_j + 1) dt\right). \end{aligned}$$

Since  $1 - a \geq 1$  and  $Re(\mu_j) \geq 0$ , as  $n \rightarrow \infty$ , we have

$$\left| ((1-2a)\lambda_j)^{e^n} \zeta(e^n, \lambda_j(1-a) + \mu_j + 1) \right| = o(1).$$

Using the functional equation for the gamma function with the reflection formula, we get

$$\begin{aligned} n! \int_{-\infty}^{+\infty} \left| \frac{\Gamma(e^n + it - n)}{\Gamma(e^n + it + 1)} \right| dt &\leq 2n! \int_0^{+\infty} \frac{dt}{((e^n - n)^2 + t^2)^{n/2}} \\ &= \frac{2n!}{(e^n - n)^{n-1}} \int_0^{+\infty} \frac{dt}{(1 + t^2)^{n/2}} \\ &\leq \frac{(n+1)^{n+1} e^{-(n+1)}}{(e^n - n)^{n-1}} \\ &\leq e^{-n}. \end{aligned}$$

Then

$$S_{Arch}(n, r) = (-1)^{n-1} n! (Res_{s=0}J_r(s) + Res_{s=1}J_r(s)) + O(1).$$

Since  $\zeta(0, x) = -x + 1/2$ , we have

$$(-1)^{n-1} n! Res_{s=0}J_r(s) = -\zeta(0, \lambda_j(1-a) + \mu_j + 1) = \lambda_j(1-a) + \mu_j + \frac{1}{2}.$$

Using Laurent series representations of the factors appearing in  $J_r(s)$ , we find

$$(-1)^{n-1} n! Res_{s=1}J_r(s) = n \left( \frac{1}{2} - a \right) \left( \frac{\Gamma'}{\Gamma}(n) - \frac{\Gamma'}{\Gamma}(\lambda_j(1-a) + \mu_j + 1) + \log \left( \frac{1}{2} - a \right) + \gamma - 1 \right).$$

The following formula

$$\frac{\Gamma'}{\Gamma}(n) = \log n - \frac{1}{2n} - \sum_{i=1}^K \frac{B_{2k}}{2k} n^{-2k} + O_K(n^{-1-2k}),$$

as  $n \rightarrow +\infty$ , yields to

$$\begin{aligned} (-1)^{n-1} n! \text{Res}_{s=1} J_r(s) &= n \left( \frac{1}{2} - a \right) \log n \\ &\quad + n \left( \frac{1}{2} - a \right) \left( -\frac{\Gamma'}{\Gamma}(\lambda_j(1-a) + \mu_j + 1) + \log \left( \frac{1}{2} - a \right) + \gamma - 1 \right) \\ &\quad - \frac{1}{4}(1-2a) - \left( \frac{1}{2} - a \right) \sum_{i=1}^K \frac{B_{2k}}{2k} n^{-2k} + O_K(n^{-1-2k}), \end{aligned}$$

as  $n \rightarrow +\infty$ . Now, from (23) and (28), we obtain

$$\begin{aligned} S_{Arch} &= \left( \frac{1}{2} - a \right) d_F n \log n \\ &\quad + n(1-2a) \left\{ \frac{d_F}{2} [\gamma - 1 - \log(1-2a)] + \frac{1}{2} \log(\lambda Q_F)^2 + C_F(a) \right\} + O(1), \end{aligned} \tag{30}$$

where

$$C_F(a) = \sum_{j=1}^r \sum_{k=0}^{+\infty} (a\lambda_j)^k \zeta(k, \lambda_j + \mu_j), \quad \lambda = \prod_{j=1}^r \lambda_j^{2\lambda_j}$$

and  $\gamma$  is the Euler constant.

Now, the non-archimedean contribution can be written in the form

$$S_{NArch} = m_F \left[ 2 - \left( 1 - \frac{2a-1}{a} \right)^n - \left( 1 + \frac{2a-1}{1-a} \right)^n \right] + \sum_{k=1}^n \binom{n}{k} (1-2a)^k \eta_{k-1}(F, 1-a).$$

A simple adaptation of the argument used in [11, Lemma 4.4] yields to that, if the Riemann hypothesis holds for  $F \in \tilde{\mathcal{S}}$ , then

$$S_{NArch} = O(\sqrt{n} \log n).$$

Finally, we obtain the same formula given in Theorem 7, for  $a < 0$ , we have

$$\begin{aligned} \lambda_F(n, a) &= (1/2 - a) d_F n \log n \\ &\quad + n(1-2a) \left\{ \frac{d_F}{2} [\gamma - 1 - \log(1-2a)] + \frac{1}{2} \log(\lambda Q_F)^2 + C_F(a) \right\} \\ &\quad + O(\sqrt{n} \log n). \end{aligned}$$

## 7 Concluding Remarks

- Suppose that the Riemann hypothesis holds and  $\lambda_F(n, a)$  are real numbers otherwise we take the real part. Then, we have

$$\lambda_F(n, a) = 2 \sum_{j=1}^{+\infty} [1 - \cos(n\theta_j)] \geq 0,$$

where  $\theta_j = \arctan\left(\frac{(1-2a)\gamma_j}{\gamma_j^2 - a^2 + a - \frac{1}{4}}\right)$  with  $\rho = \frac{1}{2} + i\gamma_j$ . We can write it as

$$\lambda_F(n, a) = 2 \int_0^{+\infty} [1 - \cos(n\theta(\gamma))] dN_F(\gamma),$$

where the lower limit just as well way be taken as  $\gamma_1$ . Integrating by parts, we get

$$\begin{aligned} \lambda_F(n, a) &= -2n \int_0^{+\infty} \sin(n\theta(\gamma)) N_F(\gamma) d\gamma \\ &\quad + [2(1 - \cos(n\theta(\gamma)) N_F(\gamma))]_0^{+\infty}. \end{aligned}$$

We made similar computation as in [5, Equations (1.4), ..., (1.9)], we get

$$\begin{aligned} \lambda_F(n, a) &= 32(1 - 2a)n \\ &\quad \times \int_0^{+\infty} \frac{\gamma N_F(\gamma)}{(4\gamma^2 - 4a^2 + 4a + 1)^2} U_{n-1}\left(\frac{4\gamma^2 - 4a^2 + 4a - 1}{4\gamma^2 - 4a^2 + 4a + 1}\right) d\gamma, \end{aligned}$$

where  $U_{n-1}$  are the Chebyshev polynomials of the second kind. Furthermore, using the following relation between the Chebyshev polynomials of the second kind and the first kind:

$$\int U_n(x) dx = \frac{1}{n+1} T_{n+1}(x),$$

we obtain

$$\lambda_F(n, a) = 2(1 - 2a) \sum_{k=1}^{+\infty} \alpha_k \left[ 1 - T_n\left(\frac{4\gamma_k^2 - 4a^2 + 4a - 1}{4\gamma_k^2 - 4a^2 + 4a + 1}\right) \right],$$

where the  $\alpha_k$  count the number of zeros with imaginary part  $\gamma_k$  including the multiplicities.

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These results can be obtained directly as in [19, §4] from the identity

$$\log \xi_F\left(\frac{\rho + a - 1}{\rho - a}\right) = \log \xi_F(0) + \sum_{n=1}^{+\infty} \lambda_F(-n, a) \frac{(z - (1 - a))^n}{n}.$$

- Using the integral formula, it is easy to refined the asymptotic formula for  $\lambda_F(n, a)$  under the Riemann hypothesis (see for example Coffey paper [5, Proposition 7]).

The author in [12] or with Omar and Ouni in [19, 20] conjectured that the Li coefficients for some  $L$ -functions are increasing. Then, a natural question is to see whether it remains true for the modified Li coefficients  $\lambda_F(n, a)$  with  $a < 0$ . Furthermore, it is an interesting question to study how the modified Li coefficients of  $L$ -functions in the Selberg class (or sub-class) are distributed as well as the sum

$$\sum_{n \leq x} \lambda_F(n, a)$$

and state a relation between a partial Li criterion which relates the partial Riemann hypothesis to the positivity of the modified Li coefficients  $\lambda_F(n, a)$  with  $a < 1/2$  up to a certain order  $T > 0$ . It is also possible to extended the argument used in [12] to the Selberg class (or sub-class) to prove that the first modified Li coefficients are increasing using the Bell polynomials without assuming the Riemann hypothesis.

These problems will be considered in a sequel to this article.

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